Variational Approach for the Lane-Emden Equation

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A variational principle for the Lane-Emden equation is established by He's semi-inverse method. Based on the established variational formulation, approximate solutions can be easily obtained by the Ritz method. The obtained solutions are in good agreement with the exact ones. The results show that the variational approach is very effective and convenient for solving the Lane-Emden equation.

Key words: Variational Principle; Lane-Emden Equation; Semi-Inverse Method.
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1. Introduction

With the rapid development of nonlinear science, various kinds of analytical methods were used to handle nonlinear problems, such as the homotopy perturbation method [1 – 5], the variational iteration method [6, 7], the parameter-expansion method [8 – 13] and the exp-function method [14, 15]; a complete review on recently developed analytical methods is given by He [16]. The variational method is an old mathematical tool, but it is not widely used for nonlinear equations possibly due to the difficulty arising in establishing its variational formulation. Now things have changed; we can use the semi-inverse method [17 – 20] to search for a needed variational principle, then the Ritz method can be effectively and conveniently used.

Many problems in the field of mathematical physics can be formulated as the Lane-Emden equation. The Lane-Emden equation is Poisson's equation for the gravitational potential of a self-gravitating, spherically symmetric polytropic fluid. This problem was studied by many researchers with various kinds of methods. In the present paper we aim at applying He's semi-inverse method to establish a variational formula of the Lane-Emden equation

$$y''(x) + \frac{2}{x}y'(x) + f(y(x)) = 0, \quad x \in [0,1], \tag{1}$$

with the initial conditions

$$y(0) = 1, \quad y'(0) = 0, \tag{2}$$

where y' is the differentiation with respect to x. This equation was studied by Yildirim and Ozis [21], using the homotopy perturbation method, and by Dehghan and Shakeri [22], using the variational iteration method. The homotopy perturbation deforms the complex equation under study to a series of linear equations easily to be solved, while the main advantage of the variational iteration method is that no variational formulation is needed and the obtained results are optimal among all possible other trial solutions.

2. Variational Formulation

Calculus of variations is an old mathematics, and was originally applied to astronomy by many famous scientists, such as Newton and Jacobi. Due to the remarkable development of computers, many problems can now be solved numerically. As a result the variational approach is rarely used in astronomy and other fields. The variational formulation in energy form has practical physical meanings, and variational-based approximate solutions are best among all possible trial functions and valid for the whole solution domain. Here we will apply the semi-inverse method [17 – 20, 23] to establish the needed variational formulation.

We re-write (1) in the form

$$xy''(x) + 2y'(x) + xf(y(x)) = 0, \quad x \in [0,1]. \tag{3}$$

According to the semi-inverse method, we construct a trial-Lagrangian in the form [17]

$$L = -\frac{1}{2}g(x)y'(x)^2 + g(x)F(y(x)), \tag{4}$$

where $g(x)$ is an unknown function of x, and $F(y)$ is an unknown function of y.

The Euler-Lagrange equation of the above Lagrangian reads

$$g(x)y''(x) + g'(x)y'(x) + g(x)\frac{\partial F}{\partial y} = 0. \tag{5}$$

Equation (5) should be equivalent to (1); thus we set

$$\frac{g(x)}{g'(x)} = \frac{x}{2} \tag{6}$$

and

$$\frac{\partial F}{\partial y} = f, \tag{7}$$

from which the unknown $g(x)$ and $F(y(x))$ can be determined as follows:

$$g(x) = x^2, \tag{8}$$

$$F(y) = \int_0^y f(s)\mathrm{d}s. \tag{9}$$

The Lagrangian is identified as

$$L = -\frac{1}{2}x^2 y'(x)^2 + x^2 F(y), \tag{10}$$

where F is defined by (7).

Finally we obtain the following variational principle:

$$J = \int_0^1 \left[-\frac{1}{2}x^2 y'(x)^2 + x^2 F(y)\right] \mathrm{d}x. \tag{11}$$

It is easy to prove that the stationary condition of the above functional, (11), is equivalent to (3).

3. Analytical Solution

We consider the case $f(y) = y^n$, where $n = 0, 1, 5$; for easy comparison, the exact solutions are [22]

$$n = 0, \quad y(x) = 1 - \frac{x^2}{6}, \tag{12}$$

$$n = 1, \quad y(x) = \frac{\sin(x)}{x}, \tag{13}$$

$$n = 5, \quad y(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2}. \tag{14}$$

For $f(y) = y^n$, the variational principle can be written as follows:

$$J = \int_0^1 \left[-\frac{1}{2}x^2 y'^2 + \frac{x^2}{n+1}y^{n+1}\right] \mathrm{d}x. \tag{15}$$

Case 1. $n = 0$.

Choosing a trial function in the form

$$y(x) = 1 + (\beta - 1 - \gamma)x + \gamma x^2, \tag{16}$$

where β and γ are unknown constants to be further determined.

Substituting (16) into (15) yields

$$\begin{aligned} J_{n=0} = \int_0^1 \Big\{ &-\frac{1}{2}x^2 [(\beta - 1 - \gamma) + 2\gamma x]^2 \\ &+ x^2 \left[1 + (\beta - 1 - \gamma)x + \gamma x^2\right] \Big\} \mathrm{d}x \\ = &-\frac{1}{15}\gamma^2 + \frac{7}{60}\gamma - \frac{1}{6}\beta\gamma - \frac{1}{6}\beta^2 + \frac{7}{12}\beta - \frac{1}{12}. \end{aligned} \tag{17}$$

Making J stationary with respect to γ results in

$$\frac{\partial J_{n=0}}{\partial \gamma} = -\frac{2}{15}\gamma - \frac{1}{6}\beta + \frac{7}{60} = 0. \tag{18a}$$

Considering the initial condition $y'(0) = 0$, we can easily obtain the result

$$\gamma = \beta - 1. \tag{18b}$$

From (18a) and (18b) we can easily obtain the following result:

$$\beta = \frac{5}{6}, \quad \gamma = -\frac{1}{6}.$$

We, therefore, obtain the exact solution of the Lane-Emden equation.

Case 2. $n = 1$.

Substituting (16) into (15) yields

$$\begin{aligned} J_{n=1} = \int_0^1 \Big\{ &-\frac{1}{2}x^2 [(\beta - 1 - \gamma) + 2\gamma x]^2 + \frac{1}{2}x^2 \\ &\left[1 + (\beta - 1 - \gamma)x + \gamma x^2\right]^2 \Big\} \mathrm{d}x \\ = &\frac{13}{210}\gamma^2 - \frac{1}{5}\beta\gamma + \frac{3}{20}\gamma - \frac{1}{15}\beta^2 + \frac{23}{60}\beta - \frac{3}{20}. \end{aligned} \tag{19}$$

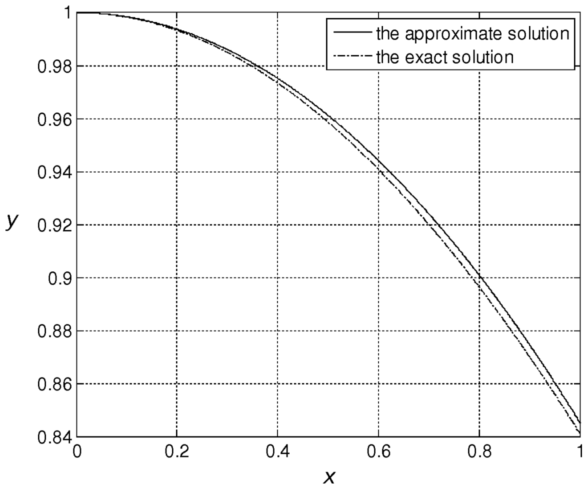

Fig. 1. Comparison of the first-order approximate solution (20) with the exact one, (13).

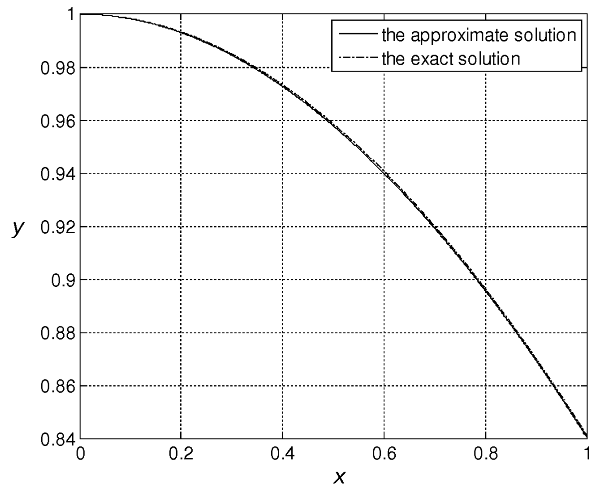

Fig. 2. Comparison of the second-order approximate solution (23) with the exact one, (13).

Making J stationary with respect to γ results in

$$\frac{\partial J_{n=1}}{\partial \gamma} = -\frac{13}{105}\gamma - \frac{1}{5}\beta + \frac{3}{20} = 0. \tag{19a}$$

Similarly considering the initial condition $\gamma'(0) = 0$, we obtain

$$\gamma = \beta - 1. \tag{19b}$$

From (19a) and (19b) we can easily obtain the following result:

$$\beta = \frac{115}{136}, \quad \gamma = \frac{21}{136}.$$

We, therefore, obtain the approximate solution

$$y(x) = 1 - \frac{21}{136}x^2. \tag{20}$$

Comparison of the approximate solution (20) with the exact solution (13) is illustrated in Fig. 1, showing a good agreement. Of course we can obtain even higher accurate solutions without any difficulty.

For example, supposing that the solution can be expressed in the form

$$y(x) = 1 + (\beta - 1 - \gamma - \eta)x + \gamma x^2 + \eta x^3, \tag{21}$$

where γ, η are unknown constants to be further determined.

Substituting (21) into (15) yields

$$\begin{aligned} J_{n=1} = \int_0^1 \Big\{ & -\frac{1}{2}x^2\left[(\beta - 1 - \gamma - \eta) + 2\gamma x + 3\eta x^2\right]^2 \\ & + \frac{1}{2}x^2\left[1 + (\beta - 1 - \gamma - \eta)x + \gamma x^2 + \eta x^3\right]^2 \Big\} \mathrm{d}x \\ = & -\frac{61}{280}\gamma\eta - \frac{13}{210}\gamma^2 - \frac{62}{315}\eta^2 - \frac{3}{20} + \frac{3}{20}\gamma \\ & + \frac{101}{420}\eta - \frac{34}{105}\beta\eta - \frac{1}{5}\beta\gamma - \frac{1}{15}\beta^2 + \frac{23}{60}\beta. \end{aligned} \tag{22}$$

Making J stationary with respect to γ and η, respectively, results in

$$\frac{\partial J_{n=1}}{\partial \gamma} = -\frac{61}{280}\eta - \frac{13}{105}\gamma - \frac{1}{5}\beta + \frac{3}{20} = 0, \tag{22a}$$

$$\frac{\partial J_{n=1}}{\partial \eta} = -\frac{124}{315}\eta - \frac{61}{280}\gamma - \frac{34}{105}\beta + \frac{101}{420} = 0. \tag{22b}$$

Considering the initial condition $y'(0) = 0$, we can easily obtain the result

$$\gamma + \eta = \beta - 1. \tag{22c}$$

From (22a) – (22c) we can easily obtain the following result:

$$\beta = \frac{10645}{12661}, \quad \gamma = \frac{2226}{12661}, \quad \eta = \frac{210}{12661}.$$

We, therefore, obtain the approximate solution

$$y(x) = 1 - \frac{2226}{12661}x^2 + \frac{210}{12661}x^3. \tag{23}$$

Comparison of the approximate solution (23) with the exact solution (13) is illustrated in Fig. 2, showing an remarkable agreement. Accuracy can be further improved if the solution procedure continues to higher order.

Case 3. $n = 5$.

Substituting (21) into (15) yields

$$J_{n=5} = \int_0^1 \left\{ -\frac{1}{2}x^2\left[(\beta-1-\gamma-\eta)+2\gamma x+3\eta x^2\right]^2 + \frac{1}{6}x^2\left[1+(\beta-1-\gamma-\eta)x+\gamma x^2+\eta x^3\right]^6 \right\} \mathrm{d}x. \quad (24)$$

Similarly to Case 2, we can easily obtain the following result:

$$\beta = 0.865349, \quad \gamma = -0.187954, \quad \eta = 0.053303.$$

We, therefore, obtain the approximate solution

$$y(x) = 1 - 0.187954x^2 + 0.053303x^3. \quad (25)$$

Comparison of the approximate solution (25) with the exact solution (14) is illustrated in Fig. 3, showing a remarkable agreement.

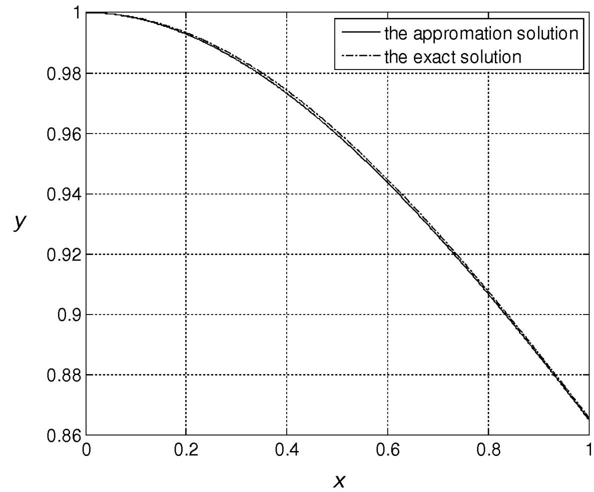

Fig. 3. Comparison of the first-order approximate solution (25) with the exact one, (14).

4. Conclusion

We obtain a variational formula for the discussed problem by He's semi-inverse method, which is proven to be a very effective and convenient way to search for the needed variational principles for the Lane-Emden equation. Based on the obtained variational formula, we can easily obtain approximate solutions by the Ritz method. The obtained solutions are in good agreement with the exact ones. The results show that the variational approach is very effective and convenient for solving the Lane-Emden equation.

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